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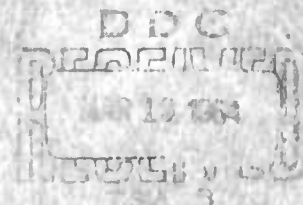
MEMORANDUM
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MARCH 1964

INCIDENCE MATRICES
AND INTERVAL GRAPHS

D. R. Fulkerson and O. A. Gross

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The RAND Corporation
1700 MAIN ST · SANTA MONICA · CALIFORNIA

PREFACE

Many problems in pure and applied mathematics depend on properties of matrices of 0's and 1's, i.e., rectangular arrays of numbers, each 0 or 1. The present Memorandum shows how a fundamental problem in genetics can be expressed and solved in terms of such matrices. The problem is to determine whether data from suitable experiments is con- sistent with the assumption that the subelements of genes are linked together in linear order.

SUMMARY

According to present genetic theory, the fine structure of genes consists of linearly ordered elements. A mutant gene is obtained by alteration of some connected portion of this structure. By examining data obtained from suitable experiments, it can be determined whether or not the blemished portions of two mutant genes intersect or not, and thus intersection data for a large number of mutants can be represented as an undirected graph. If this graph is an "interval graph," then the observed data is consistent with a linear model of the gene.

The problem of determining when a graph is an interval graph is a special case of the following problem concerning (0, 1)-matrices: When can the rows of such a matrix be permuted so as to make the 1's in each column appear consecutively? A complete theory is obtained for this latter problem, culminating in a decomposition theorem which leads to a rapid algorithm for deciding the question, and for constructing the desired permutation when one exists.

CONTENTS

INCIDENCE MATRICES AND INTERVAL GRAPHS

1. INTRODUCTION

Let $A = (a_{ij})$ be an m by n matrix whose entries a_{ij} are all either 0 or 1. The matrix A may be regarded as the incidence matrix of elements e_1, e_2, ..., e_m vs. sets S_1, S_2, ..., S_n; that is, $a_{ij} = 0$ or 1 according as e_i is not or is a member of S_j. For certain applications, one of which will be discussed below, it is of interest to know whether or not one can order the elements in such a way that each set S_j consists of elements that appear consecutively in the ordering. In terms of the incidence matrix A, the question is whether there is an m by m permutation matrix P such that the 1's in each column of PA occur in consecutive positions. We shall describe a computationally efficient method of answering this question, and of determining such a P when one exists.

Given a family of sets S_1, S_2, ..., S_n, one can form the <u>intersection graph</u> of the family by associating a vertex of the graph with each set and joining two distinct vertices with an edge if their corresponding sets have a nonempty intersection. Conversely, any finite graph can of course be viewed as the intersection graph of a family of sets (in many ways). If each set can be taken as an interval on the real line, the graph is called an <u>interval graph</u>. Interval graphs have been investigated in [3, 4, 5].

The problem posed above is closely related to that of determining whether a given graph is an interval graph. Necessary and sufficient conditions for this are known [3, 5]. But our problem appears to be more general, and our approach is quite different.

The study of interval graphs was stimulated in part by a biological application concerning the fine structure of genes [1, 3]. A basic problem, posed in [1], is to decide whether or not the subelements of genes are linked together in a linear order. A way of approaching this problem is also described in [1]. Briefly, it is as follows. For certain microorganisms, there are a standard form and mutants, the latter arising from the standard form by alteration of some connected part of the genetic structure. Experiments can be devised for determining whether the blemished parts of two mutant genes intersect or not. Thus the mathematical problem becomes: Given a large number of mutants together with information as to when the blemished portions of pairs of mutants intersect, to decide whether this information is compatible with a linear model of the gene or not. One example, analyzed in [1], shows intersection data for 145 mutants of a certain virus, for which it was found that a linear model was adequate.

In attacking the combinatorial problem posed at the beginning of this section, it does not suffice to consider just the intersection graph of the sets S_1, S_2, ..., S_n. Instead, we focus attention primarily on a more restricted graph, the _overlap graph_ (Sec. 3), two sets being said here to overlap if they have a nonempty intersection which is properly included in each. The connected components of the overlap graph partition the m by n incidence matrix A into m-rowed submatrices A_1, A_2, ..., A_p. Thus we can take $A = (A_1, A_2, ..., A_p)$, where each A_k corresponds to a component of the overlap graph. Then our main theorem (Theorem 4.1) asserts that if there are permutations P_k, $k = 1, 2, ..., p$, such that $P_k A_k$ has consecutive 1's in each column, there is a permutation P such that PA has this property also. This somewhat surprising result, coupled with the fact that one can describe a simple and direct construction (Sec. 5) for testing whether such P_k exists, provides an answer to the existence question for a general incidence matrix A.

2. A BASIC THEOREM

Let A be a (0, 1)—matrix. We say that A has the _consecutive_ 1's _property_ (for columns) provided there is a permutation matrix P such that the 1's in each column of PA occur consecutively. We note to begin with that the intersection graph of A does not contain enough information to decide whether A has the property or not.

For example, the matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

have the same intersection graph, a triangle, but A_2 does not have the consecutive 1's property. The first question that naturally comes up, then, is how much information about A is needed to decide whether it has the property or not. Do we need to know A itself, or will something less suffice? Theorem 2.1 below provides a partial answer to this question; it shows that a knowledge of the matrix $A^T A$ is enough. Here A^T denotes the transpose of A.

Theorem 2.1. Let A and B be (0, 1)-matrices satisfying

$$(2.1) \qquad A^T A = B^T B .$$

Then either both A and B have the consecutive 1's property or neither does. Moreover, if A and B have the same number of rows and A has the consecutive 1's property, then there is a permutation P such that B = PA.

Proof. The first part of the theorem follows trivially from the second. For assume (2.1) and let A be m by n, B be m' by n, with $m > m'$. Then we may adjoin m—m' rows of 0's to B, thereby obtaining an m by n matrix C satisfying $A^T A = C^T C$.

The second assertion of the theorem then implies $C = PA$, for some permutation P. Consequently both A and C (hence both A and B) have the property or neither does.

Let A and B be m by n. The second part of the theorem is obviously valid for $n = 1$, and we proceed by induction on n. Suppress the first column vector a of A and the first column vector b of B, and call the resulting matrices A_1 and B_1. Thus

$$(2.2) \qquad A = (a, A_1) ,$$

$$(2.3) \qquad B = (b, B_1) .$$

Clearly A_1 has the consecutive 1's property and $A_1^T A_1 = B_1^T B_1$. Hence, by the induction hypothesis, we can permute the rows of B to obtain a matrix $\tilde{B}$ satisfying

$$(2.4) \qquad \tilde{B} = (\tilde{b}, A_1) ,$$

the column vector $\tilde{b}$ being a permutation of b.

We shall finish the proof by showing that, corresponding to each row vector of $\tilde{B}$, there is an equal row vector of A. We begin by noting that if the column vectors a of (2.2) and $\tilde{b}$ of (2.4) have common 1's (common 0's), the corresponding rows of A and $\tilde{B}$ are equal and can be paired off against each other. Having done this, consider the submatrices

of A and $\widetilde{B}$ consisting of the remaining rows. Call these
matrices A* and B*. Thus we may write

(2.5) $A^* = (a^*, A_1^*)$,

(2.6) $B^* = (b^*, A_1^*)$.

Note that A* inherits the consecutive 1's property from
A, since suppression of rows does not destroy the property,
and hence we can permute the rows of A* to obtain a matrix

(2.7) $\overline{A} = (\overline{a}, \overline{A}_1)$

in which the 1's in each column occur consecutively.
Applying the same permutation to B* yields

(2.8) $\overline{B} = (\overline{b}, \overline{A}_1)$.

Now $\overline{a}$ and $\overline{b}$ are complementary (0, 1)–vectors, that is, $\overline{b}$
is obtained from $\overline{a}$ by interchanging 0's and 1's. We also
have

(2.9) $\overline{A}^T\overline{A} = \overline{B}^T\overline{B}$,

since $\overline{A}$ and $\overline{B}$ are obtained from A and B, respectively,
by suppressing certain equal rows and permuting rows.
In particular, calculating the inner product of $\overline{a}$ with

itself and $\overline{b}$ with itself, we have

(2.10) $\overline{a}\cdot\overline{a} = \overline{b}\cdot\overline{b}$,

and hence $\overline{a}$ and $\overline{b}$ have the same number of 1's (hence also of 0's), so that the number of 1's in $\overline{a}$ is equal to the number of 0's in $\overline{a}$. Thus $\overline{A}$ and $\overline{B}$ have the following appearance:

$$(2.11) \quad \overline{A} = \begin{bmatrix} 0 \\ \vdots & E \\ 0 \\ \hline 1 \\ \vdots & F \\ 1 \\ \hline 0 \\ \vdots & G \\ 0 \end{bmatrix} \quad , \quad \overline{B} = \begin{bmatrix} 1 \\ \vdots & E \\ 1 \\ \hline 0 \\ \vdots & F \\ 0 \\ \hline 1 \\ \vdots & G \\ 1 \end{bmatrix} \quad .$$

The matrices E and G in (2.11) have together the same number of rows that F does, and we shall show that the rows of $\overline{A}$ corresponding to F (corresponding to E and G) can be paired with equal rows of $\overline{B}$ corresponding to E and G (corresponding to F).

To this end, let E and G have k and ℓ rows, respectively, and let R_p denote the p by p permutation matrix which reverses order,

$$
(2.12) \qquad R_p = \begin{bmatrix} & & & 1 \\ & 0 & \cdot & \\ & \cdot & \cdot & \\ & \cdot & \cdot & 0 \\ 1 & & & \end{bmatrix} \quad .
$$

We assert that

$$
(2.13) \qquad F = \begin{bmatrix} R_k E \\ \\ R_\ell G \end{bmatrix} \quad .
$$

For consider an arbitrary column vector c of $\overline{A}_1$ and write

$$
(2.14) \qquad c = \begin{bmatrix} e \\ f \\ g \end{bmatrix} ,
$$

corresponding to the partition (2.11) of $\overline{A}_1$. From (2.9) we have

$$
\overline{a} \cdot c = \overline{b} \cdot c ,
$$

and thus the number of 1's in f is equal to the number of 1's in e and g. It follows from this and the fact that the 1's in c occur consecutively that

$$(2.15) \qquad f = \begin{bmatrix} R_k e \\ R_\ell g \end{bmatrix} .$$

This establishes (2.13) and finishes the proof of Theorem 2.1.
We note the following corollary of Theorem 2.1.

Corollary 2.2. Let A and B be (0, 1)—matrices
satisfying (2.1). If A has the consecutive 1's property
and has no rows of 0's, then there is a permutation P
such that

$$(2.16) \qquad PB = \begin{bmatrix} A \\ 0 \end{bmatrix} .$$

Proof. Let A be m by n and let B be m' by n. Suppose
m' < m. Then we may adjoin m—m' rows of 0's to B, obtaining
a matrix C satisfying $A^T A = C^T C$. By Theorem 2.1, C is a
row permutation of A, violating the assumption that A has
no zero rows. Consequently m' $\geq$ m. The conclusion now
follows by adjoining m'—m rows of 0's to A.

It is of course not true in general that equation
(2.1) implies B = PA for m by n (0, 1)—matrices A and B.
A simple example is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

It would be interesting to know conditions on $A^T A$ in order that A have the consecutive 1's property. Although we have not been able to give a complete answer to this question in the sense of finding some nice set of necessary and sufficient conditions, we shall establish a theorem in Sec. 4 which reduces the question to the consideration of (0, 1)—matrices having connected overlap graphs. Section 5 then provides a simple construction for testing whether or not such a matrix has the property.

3. THE OVERLAP GRAPH AND COMPONENT GRAPH

We rephrase the definition of "overlap" in terms of inner products of (0, 1)—vectors. Let a and b be (0, 1)—vectors having m components. Their inner product $a \cdot b$ satisfies

$$(3.1) \qquad 0 \leq a \cdot b \leq \min (a \cdot a, \ b \cdot b) \ .$$

If strict inequality holds throughout (3.1), that is, if $a \cdot b$ satisfies

$$(3.2) \qquad 0 < a \cdot b < \min (a \cdot a, \ b \cdot b) \ ,$$

we say that a and b overlap. We also say that a and b are disjoint if

$$(3.3) \qquad a \cdot b = 0 \ ,$$

and that a <u>contains</u> b if

$$(3.4) \qquad a \cdot b = b \cdot b .$$

Now let A be an m by n $(0, 1)$—matrix having column vectors $a_1, a_2, \ldots, a_n$. It is convenient, and imposes no loss of generality in studying the consecutive 1's property, to assume that $a_j \neq 0$, $j = 1, 2, \ldots, n$, and that $a_i \neq a_j$ for $i \neq j$. Henceforth we frequently make these assumptions and refer to such an A as <u>proper</u>.

There are various graphs one can associate with a $(0, 1)$—matrix A that are meaningful insofar as the consecutive 1's property is concerned. For instance, we can take vertices $x_1, x_2, \ldots, x_n$ corresponding to the columns $a_1, a_2, \ldots, a_n$ of A, and put in the following edges, some being directed, others undirected: an edge $[x_i, x_j]$ directed from x_i to x_j if a_i contains a_j; an undirected edge (x_i, x_j) joining x_i and x_j if a_i and a_j overlap. (If we go on to make this a "weighted" graph by associating with each edge (vertex) the appropriate inner product, then Theorem 2.1 shows that we have included sufficient information to decide whether A has the consecutive 1's property.) We shall not, however, deal primarily with this full graph $\mathscr{F} = \mathscr{F}(A)$, but shall instead work mostly with two graphs derived from it, one of these being an undirected graph, the other a directed graph.

The first of these is the graph obtained from $\mathcal{F}$ by including only the undirected edges (x_i, x_j) corresponding to overlapping column vectors a_i and a_j. We call this the overlap graph of A and denote it by $\mathcal{G} = \mathcal{G}(A)$.

The second of these graphs is obtained from $\mathcal{F}$ as follows: Let X_1, X_2, ..., X_p be the connected components of $\mathcal{G}(A)$, considered as vertices of a graph $\mathcal{D} = \mathcal{D}(A)$. Direct an edge $[X_i, X_j]$ from vertex X_i to vertex X_j if there is an x in X_i and a y in X_j such that $[x, y]$ is an edge from x to y in $\mathcal{F}$. We call $\mathcal{D}$ the component graph of A. The component graph $\mathcal{D}$ is obtained by condensing in $\mathcal{F}$ the vertices of a connected component of $\mathcal{G}$ to a single vertex.

We give an example illustrating these concepts. Let

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} .
$$

Thus

$$A^T A = \begin{bmatrix} 6 & 5 & 5 & 0 & 0 & 2 & 2 & 3 \\ 5 & 8 & 5 & 2 & 2 & 2 & 2 & 3 \\ 5 & 5 & 5 & 0 & 0 & 2 & 2 & 3 \\ 0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 2 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 1 & 2 & 0 \\ 3 & 3 & 3 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} .$$

The graphs $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{D}$ for A are shown in Figs. 3.1, 3.2, and 3.3, respectively.

Notice in the example that the component graph $\mathcal{D}$ is acyclic (has no directed cycles) and transitive (if [X, Y] and [Y, Z] are edges, then [X, Z] is an edge). We now prove that this is generally so.

Theorem 3.1. The component graph $\mathcal{D}$(A) of a proper (0, 1)—matrix A is acyclic and transitive.

Proof. We first show that $\mathcal{D}$ is transitive. Let X, Y, Z be components of $\mathcal{G}$ (vertices of $\mathcal{D}$) such that [X, Y] and [Y, Z] are edges in $\mathcal{D}$. Hence there are vertices x, y, y', z of $\mathcal{F}$ with x in X, y and y' in Y, and z in Z, such that [x, y] and [y', z] are directed edges of $\mathcal{F}$. Moreover, since y and y' are in the same component of $\mathcal{G}$, there is a chain (y, y_1), (y_1, y_2), ..., (y_k, y') of undirected edges joining y to y' in $\mathcal{G}$ (hence in $\mathcal{F}$). Thus the matrix A has corresponding columns a, b, b', c and
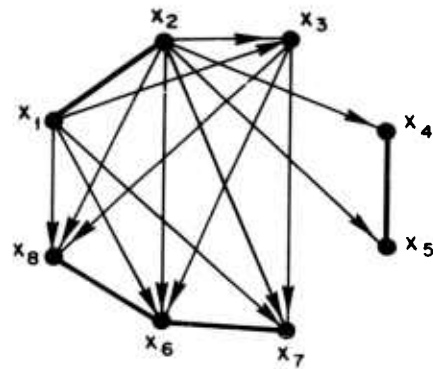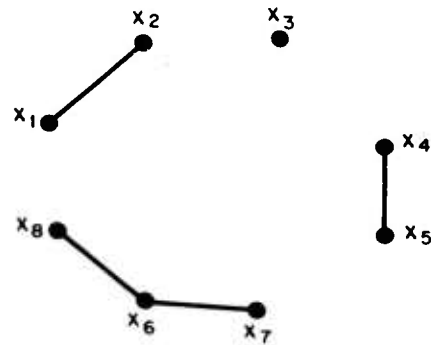
Fig. 3.1—The full graph $\mathscr{G}(A)$
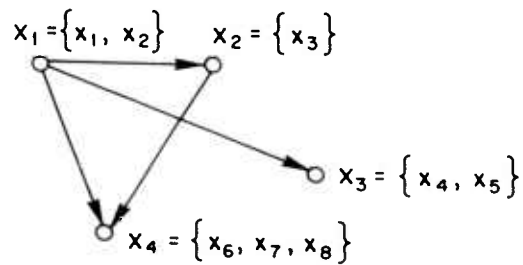

Fig. 3.2—The overlap graph $\mathcal{G}(A)$

Fig. 3.3—The component graph $\mathfrak{D}(A)$

b_1, b_2, ..., b_k such that a contains b, b' contains c,
b and b_1 overlap, b_1 and b_2 overlap, ..., b_k and b' over-
lap. But if a contains b and b overlaps b_1, then either
a contains b_1 or a overlaps b_1. The latter of these
alternatives is impossible, for otherwise x and y_1 would
be in the same component of the overlap graph, contrary
to choice. Thus a contains b_1. We may now repeat the
argument with a, b_1, b_2 in place of a, b, b_1 to show that
a contains b_2, and so on, finally deducing that a contains
b'. Since b' contains c, we see that a contains c, and
hence [x, z] is a directed edge in $\mathcal{F}$. Consequently
[X, Z] is a directed edge in $\mathcal{D}$, and $\mathcal{D}$ is transitive.

To show that $\mathcal{D}$ is acyclic, it suffices, since $\mathcal{D}$
is transitive, to show that both [X, Y] and [Y, X] cannot
be edges of $\mathcal{D}$. Suppose they were edges of $\mathcal{D}$. Then, as
above, the matrix A would have columns a_1, a_2, ..., a_k
and b_1, b_2, ..., b_ℓ with the following relations holding:
a_1 contains b_1; b_ℓ contains a_k; a_1 overlaps a_2, ...; a_{k-1}
overlaps a_k; b_1 overlaps b_2, ...; $b_{\ell-1}$ overlaps b_ℓ; and
no a_i overlaps any b_j. It then follows as above that a_1
contains b_ℓ. Similarly we deduce that b_ℓ contains a_1.
Thus $a_1 = b_\ell$. Since a_1 and b_ℓ are necessarily distinct
columns of A, this violates our assumption that A has no
pair of equal columns. Consequently $\mathcal{D}$ is acyclic. This
completes the proof of Theorem 3.1.

Theorem 3.1 shows that $\mathcal{D}$ is the graph of a partial
ordering. This partial ordering is special in the sense

that an element can have at most one immediate predecessor,
as Theorem 3.2 below shows. Consequently, if we omit from
$\mathcal{D}$ every edge whose existence is implied by transitivity,
the resulting graph is simply a collection of rooted trees.
(For example, in Fig. 3.3, omitting the edge $[X_1, X_4]$ produces
a single tree with root X_1.)

Theorem 3.2. Each vertex of the component graph
$\mathcal{D}(A)$ of a proper (0, 1)–matrix A has at most one immediate
predecessor.

Proof. It suffices to show that if $[X, Z]$ and $[Y, Z]$
are edges of $\mathcal{D}$, then either $[X, Y]$ or $[Y, X]$ is an edge.
Thus, let $[X, Z]$ and $[Y, Z]$ be edges of $\mathcal{D}$. It follows
that A has columns a, b, c_1, c_2, ..., c_k with the following
relations holding: a contains c_1, b contains c_k, and
successive pairs of c's overlap. Thus a contains c_k.
Since b also contains c_k and a and b do not overlap, then
either a contains b or b contains a. Consequently, either
$[X, Y]$ or $[Y, X]$ is an edge of $\mathcal{D}$.

In Sec. 4 we shall use the structure of the component
graph to prove the decomposition theorem mentioned in
Sec. 1. This structure will also be exploited later in
developing a complete algorithm for arranging the rows
of a matrix A to make its 1's appear consecutively in
each column, when this is possible. For these purposes,
we note here another fact about the full graph $\mathcal{F}(A)$ that
has not been stated explicitly, although its proof is

contained in that of Theorem 3.1.

Theorem 3.3. Let X and Y be components of the overlap graph $\mathcal{G}(A)$ of a proper (0, 1)—matrix A, such that there is an x in X and y in Y with [x, y] an edge of $\mathcal{F}(A)$. Then [x, y'] is an edge of $\mathcal{F}(A)$ for arbitrary y' in Y.

Theorem 3.3 shows that each column of A corresponding to Y plays exactly the same role with respect to columns of A corresponding to X. That is, each "Y—column" will be contained in certain of the "X—columns" and disjoint from the others. This pattern is the same for every Y—column.

4. THE DECOMPOSITION THEOREM

For an arbitrary (0, 1)—matrix A, we can rearrange columns if necessary and write

$$(4.1) \qquad A = (A_1, A_2, \ldots, A_p) ,$$

where each submatrix A_k, $k = 1, 2, \ldots, p$, corresponds to a component X_k of the overlap graph $\mathcal{G}(A)$. We term (4.1) an overlap decomposition of A. It is of course unique, apart from the ordering of the submatrices. We refer to these submatrices as overlap components of A, or briefly, components of A. If A has just one component, we say that A is connected.

Theorem 4.1. <u>A</u> (0, 1)—<u>matrix</u> A <u>has the consecutive</u>
<u>1's property if and only if each of its components has</u>
<u>the property</u>.

Proof. The necessity is obvious. To prove sufficiency,
we may assume that A is proper and proceed by induction on
the number p of components of A, the case p = 1 being
trivial.

Before going to the induction step, we first remark
that if a matrix has the consecutive 1's property and has
equal rows, such rows can be brought together in a permuted
form which makes the 1's in each column consecutive. To
see this, it suffices to observe two things. First, if a
row is deleted from a matrix having the consecutive 1's
property, the resulting matrix has the property. On the
other hand, if a deleted row is one of a pair of equal
rows, then it can be reinserted adjacent to its counter—
part in the permuted matrix, and the 1's in each column
are still consecutive.

To establish the induction step, let A be a proper
matrix having p components, each of which has the con—
secutive 1's property, and write

(4.2) $$A = (A_1, A_2, \ldots, A_p),$$

where A_p corresponds to a minimal element in the partial ordering given by $\mathcal{O}(A)$. Now delete A_p from A. By the induction assumption, the matrix

$$(4.3) \qquad A' = (A_1, A_2, \ldots, A_{p-1})$$

has the consecutive 1's property, and hence, by the above remark, there is a permutation P such that PA' has consecutive 1's in each column and like rows of PA' occur together. Consider

$$(4.4) \qquad PA = (B_1, B_2, \ldots, B_p) = B .$$

We may select a topmost and bottommost 1 in B_p, and write

$$(4.5) \qquad B = \begin{bmatrix} * & * & \cdots & * & 0 \\ \bar{B}_1 & \bar{B}_2 & \cdots & \bar{B}_{p-1} & \bar{B}_p \\ * & * & \cdots & * & 0 \end{bmatrix} ,$$

where the first and last rows of $\bar{B}_p$ each contain at least one 1. Now consider the matrix

$$(4.6) \qquad (\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_{p-1}) .$$

We assert that all rows of (4.6) are identical. To see this, observe first that since X_p is a minimal element in the partial ordering, each X_k, k = 1, 2, $\ldots$, p-1,

either dominates X_p (there is an edge from X_k to X_p in $\mathscr{O}(A)$) or is unrelated to X_p (there is no edge from X_k to X_p or from X_p to X_k). Suppose X_k is unrelated to X_p. Then, since the first and last rows of B_p contain a 1, it follows that the first and last rows of B_k are all 0's, for otherwise there would be a column of A_p and a column of A_k which are not disjoint. On the other hand, suppose X_k dominates X_p. It then follows from Theorem 3.3 that the first and last rows of B_k are equal. Hence the first and last rows of (4.6) are equal, and consequently, by the selection of P, all rows of (4.6) are equal.

We may now permute the rows of B_p to make its 1's consecutive in each column. Such a permutation merely shuffles like rows in (4.5), and thus A has the consecutive 1's property.

Any component of A which has no more than two columns obviously has the consecutive 1's property. We may thus state the following corollary of Theorem 4.1.

Corollary 4.2. If each component of a (0, 1)-matrix A has at most two columns, then A has the consecutive 1's property.

5. TESTING A CONNECTED MATRIX FOR THE CONSECUTIVE 1's PROPERTY

In this section we shall describe a construction for deciding whether a connected matrix A has the consecutive 1's property. The essential idea is not to search explicitly

for a permutation P which rearranges A, but rather to
attempt to build up a permuted form of A, column by column,
imposing necessary inner-product requirements on the columns
of the configuration.

Let a, b, c be distinct column vectors of A such that
a overlaps b and b overlaps c. If A has the consecutive
1's property, then there is a row permutation of A which
contains the configuration C_1 or C_2 described below (and
illustrated in Figs. 5.1 and 5.2) according, as

$$(5.1) \qquad a \cdot c < \min (a \cdot b, \ b \cdot c)$$

or

$$(5.2) \qquad a \cdot c \geq \min (a \cdot b, \ b \cdot c)$$

holds. The configuration C_1 is obtained by writing down
$a \cdot a$ 1's consecutively in a column, then $b \cdot b$ consecutive
1's in a second column so that these two strings of 1's
overlap (at the bottom of the first, say) on $a \cdot b$ 1's,
then a third string of $c \cdot c$ 1's, overlapping the second
string in the same way that the second overlapped the
first (at its bottom) on $b \cdot c$ 1's. The configuration C_2
differs from C_1 only by making the third string of 1's
overlap the second opposite to the manner in which the
second overlapped the first.

a	b	c
1		
1	1	
1	1	
1	1	
1	1	1
	1	1
	1	1
		1

Fig. 5.1. The Configuration C_1

a	b	c
1		1
1	1	1
1	1	1
1	1	1
1	1	
	1	
	1	

Fig. 5.2. The Configuration C_2

Note that while we have two choices in placing the
second string of 1's relative to the first (top or bottom
of the first), it is immaterial which of these we select.
Moreover, having selected one, there is no further latitude
in positioning the third string of 1's, since its position
relative to the second is determined by (5.1) or (5.2)
and the position of the second relative to the first.
After constructing the configuration C_1 or C_2, we can

then check the inner product of its first and third columns.
If this is not equal to a·c, we know immediately that A
does not have the consecutive 1's property.

Three column vectors of A which can be ordered so
that the first and second overlap and the second and third
overlap will be termed a __rigid triple__ of A. Our con-
struction for a connected A will be based on singling out
a spanning subtree of $\mathcal{Y}(A)$ and then positioning strings
of 1's, using rigid triples obtained from this tree.

Instead of describing the computation formally, we
shall consider an example which will make the process clear.
Let

$$
A =
\begin{matrix}
① & ② & ③ & ④ & ⑤ & ⑥ \\
\end{matrix}
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
$$

A spanning subtree of the overlap graph $\mathcal{Y}(A)$ is shown
in Fig. 5.3 below.
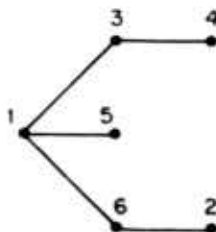
Fig. 5.3.   A Spanning Subtree of $\mathcal{Y}(A)$

On the first step, we may position the rigid triple
1, 3, 5 to obtain the configuration of Fig. 5.4 below.

```
①   ③   ⑤
          1
1        1
1        1
1    1   1
1    1
     1
```

Fig. 5.4

The inner products of this configuration agree with the
corresponding ones of A, and we proceed to add column 6,
positioning it relative to 1 and 5 (5, 1, 6 form a rigid
triple), to obtain the configuration shown in Fig. 5.5.

```
①  ③  ⑤  ⑥
        1
1       1
1       1
1   1   1
1   1       1
    1       1
```

Fig. 5.5

Checking the inner product of column 6 with all others
in the configuration shows that we have no contradiction yet,
and so we add column 4, positioning it relative to 1 and
3, to obtain the configuration of Fig. 5.6.

①	③	⑤	⑥	④
	1			
1	1			
1	1		1	
1	1	1		1
1	1		1	1
	1		1	

Fig. 5.6

Again all inner products of column 4 with preceding columns of the configuration check out properly. We then add column 2, positioning it relative to 1 and 6, obtaining the configuration of Fig. 5.7.

①	③	⑤	⑥	④	②
	1				
1	1				
1	1		1		
1	1	1		1	1
1	1		1	1	1
	1		1		

Fig. 5.7

Since the inner products of column 2 with all preceding columns in the configuration of Fig. 5.7 agree with the corresponding inner products of columns of A, Corollary 2.2 show that the matrix

$$
B = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array},
$$

obtained from Fig. 5.7 by rearranging columns and filling in 0's so that B has no zero rows, is a row permutation of A.

A complete test for a general matrix A is now clear. First determine the connected components of A. This can be easily done in such a way that corresponding spanning subtrees are automatically obtained. Simply select an arbitrary column and find all columns it overlaps; then select one of these and determine any new columns it overlaps, and so on. When no new columns can be determined, a spanning subtree for one component has been found. The configuration building process described above can then be applied to each component. (In actual computation, one would build the configuration simultaneously with the determination of a connected component.)

If one measures the efficiency of this method by calculating an upper bound on the number of times that it is necessary to compute (or look up) the inner product of two (0, 1)-vectors, it is not difficult to see that if A has n columns, such a bound is of the order $O(n^2)$.

6. FITTING THE COMPONENTS TOGETHER

If no contradiction in inner products is encountered in the construction outlined in the preceding section, one can go on to find the desired row permutation of A by fitting the various connected submatrices together in an appropriate way, using the partial ordering of components given by $\mathcal{O}(A)$. (We assume that A is proper.) The proof of Theorem 4.1 indicates how to proceed. Again we shall not describe the process in detail, but shall illustrate it with the example of Sec. 3.

The configuration—building procedure of Sec. 5 produces the matrices

$$
B_1 = \begin{bmatrix} \overset{①}{1} & \overset{②}{0} \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad
B_2 = \begin{bmatrix} \overset{③}{0} \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad
B_3 = \begin{bmatrix} \overset{④}{0} & \overset{⑤}{0} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad
B_4 = \begin{bmatrix} \overset{⑥}{0} & \overset{⑦}{0} & \overset{⑧}{0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.
$$

To fit these together, we first determine the maximal elements in $\mathcal{O}(A)$. Here there is just one, corresponding to B_1, and we have:

$$
\begin{array}{cccccccc}
\textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} & \textcircled{8}
\end{array}
$$

$$
\left[
\begin{array}{cc|ccccc}
1 & 0 & & & & & \\
1 & 1 & & & & & \\
1 & 1 & & & & & \\
1 & 1 & & & & & \\
1 & 1 & & & * & & \\
1 & 1 & & & & & \\
0 & 1 & & & & & \\
0 & 1 & & & & & \\
0 & 1 & & & & &
\end{array}
\right] .
$$

(If there were more than one maximal element, the determined part of the configuration at this point would appear as a direct sum.)

Next suppress the maximal elements in $\mathscr{O}(A)$ and look for the new maximal elements in the reduced graph. Here there are two, corresponding to B_2 and B_3. By Theorem 3.3, each column of B_2 (B_3) is contained in certain columns of B_1 and disjoint from the remaining, and this pattern is the same for all columns of B_2 (B_3). Hence it suffices to test one column of B_2 (B_3) to find this pattern. We can do this by referring to $A^{T}A$ or $\mathscr{F}(A)$. The single column of B_2 is contained in both columns of B_1, whereas columns of B_3 are disjoint from the first column of B_1 and contained in the second. We now fit in B_2 by sliding its nonzero part (the part between a topmost and bottommost 1 of B_2) to the top of the consecutive group of equal rows of B_1 consisting of (1, 1)–pairs. Similarly, we fit in B_3 by

sliding its nonzero part to the top of the consecutive group of (0, 1)—rows of B_1. This gives the configuration

$$
\begin{array}{cccccccc}
① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧
\end{array}
$$

$$
\left[
\begin{array}{cc|c|cc}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1
\end{array}
\qquad *
\right] .
$$

The new reduced component graph consists of the single element corresponding to B_4, and we see from $\mathscr{H}(A)$ that each column of B_4 is contained in columns 1, 2, 3 and is disjoint from columns 4 and 5. Consequently we slide the nonzero part of B_4 to appear at the top of the consecutive group of (1, 1, 1, 0, 0)—rows in the configuration. This yields a row permutation of A whose 1's appear consecutively in each column:

$$
\begin{array}{cccccccc}
① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \\
\end{array}
$$

$$
\left[
\begin{array}{cc|c|cc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\right] .
$$

Note that at each stage in solving the example, we
automatically had equal rows occurring together (see the
proof of Theorem 4.1) and hence did not need to permute
rows of the configuration before fitting in another piece.
It is not difficult to show inductively that this will
always be the case, provided the prescription of sliding
the nonzero part of the new piece to the top of the group
of like rows which precede it is followed. A relevant
fact in making such a proof is that the nonzero part of
the new piece contains no zero rows. This follows from
connectedness and consecutivity of 1's, and justifies our
terminology.

7. APPLICATION TO INTERVAL GRAPHS

The methods developed in preceding sections can be
applied to the problem of determining when a graph is an
interval graph. As noted in Sec. 1, various sets of
necessary and sufficient conditions that a graph be an

interval graph are known. Moreover, computational methods
based on some of these have been described [3, 5]. The
procedure we shall outline appears to be more efficient.
In addition, a specific representation in terms of intervals
can be easily produced if desired.

We begin by reviewing certain concepts and results that
will be relevant. The first of these is that of a rigid
circuit graph [2, 3]. An undirected graph* $\mathcal{G}$ is a <u>rigid
circuit graph</u> if every circuit of $\mathcal{G}$ with more than three
vertices has a chord. Here a chord is an edge not in the
circuit which joins two vertices of the circuit. It is
easy to see that an interval graph must be a rigid circuit
graph [3]. We also recall the following basic result about
rigid circuit graphs [2]: Every rigid circuit graph $\mathcal{G}$
has a simplicial vertex. Here a vertex s of $\mathcal{G}$ is <u>sim-
plicial</u> if all the neighbors of s in $\mathcal{G}$ form a simplex or
clique in $\mathcal{G}$ (i.e., each pair of neighbors of s is joined
by an edge.) Thus s together with its neighbors is also a
clique in $\mathcal{G}$.

Rigid circuit graphs have been characterized in various
ways [2, 3]. For our purposes, the following procedure
for testing the rigid circuit property will be appropriate:**

---

*All graphs considered in this section are finite and
undirected, and have no multiple edges or loops.

**This simple test has apparently not been noted before.
See [3], where a more complicated test is used.

Search for a simplicial vertex in $\mathcal{G}$; if one is found,
supress it and repeat the procedure in the reduced graph.
It follows that $\mathcal{G}$ is a rigid circuit graph if and only if
this process terminates in the deletion of all vertices
of $\mathcal{G}$. For if $\mathcal{G}$ is a rigid circuit graph, then $\mathcal{G}$ con-
tains a simplicial vertex, and deletion of vertices maintains
the rigid circuit property. Conversely, if $\mathcal{G}$ has a circuit
with more than three vertices which has no chord, then no
vertex of this circuit can be deleted in the process.

Figures 7.1 and 7.2 show examples of rigid circuit
graphs, the vertices being numbered in such a way that
successively suppressing the next vertex in the numbering
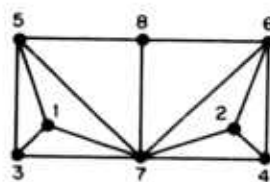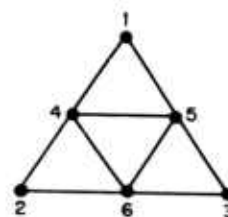fulfills the test.

Fig. 7.1

Fig. 7.2

We shall return to rigid circuit graphs in a moment, but at this point we need to introduce a certain incidence matrix which can be associated with an arbitrary graph $\mathcal{G}$. We call this matrix the <u>dominant clique</u> vs. <u>vertex</u> matrix. As will be obvious, it specifies $\mathcal{G}$ completely. Let $\mathcal{G}$ be any graph. We can consider the family of all sets of vertices which form cliques in $\mathcal{G}$ and partially order these by set inclusion. The maximal elements in this partial ordering will be termed the <u>dominant cliques</u> of $\mathcal{G}$. Since two vertices of $\mathcal{G}$ are joined by an edge if and only if they belong to some dominant clique, the dominant clique vs. vertex incidence matrix characterizes $\mathcal{G}$.

Such incidence matrices for the graphs of Figs. 7.1 and 7.2 are shown in Figs. 7.3 and 7.4, respectively.

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\qquad
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
$$

Fig. 7.3   Fig. 7.4

For rigid circuit graphs, the determination of the dominant cliques can be carried out in conjunction with the test for the rigid circuit property described above. Upon deleting a simplicial vertex, simply list it together with its neighbors in the reduced graph. If the original graph

$\mathcal{G}$ has n vertices, this yields a set of n cliques. The dominant ones of these n cliques are the dominant cliques of $\mathcal{G}$. For these are certainly cliques of $\mathcal{G}$. Moreover, if $\mathcal{C}$ is an arbitrary clique in $\mathcal{G}$, consider the first vertex of $\mathcal{C}$ which is deleted in the process. At this stage we list a clique $\mathcal{C}'$ that includes $\mathcal{C}$.

The theory and methods we have developed for studying the consecutive 1's property can now be applied to interval graphs via Theorem 7.1 below.

Theorem 7.1. A graph $\mathcal{G}$ is an interval graph if and only if the dominant clique vs. vertex incidence matrix of $\mathcal{G}$ has the consecutive 1's property.

Proof. Let D be the dominant clique vs. vertex incidence matrix of $\mathcal{G}$, so arranged that the 1's in each column occur consecutively. Suppose the first and last 1's in the j-th column of D occur in positions f_j and ℓ_j, respectively. Then $\mathcal{G}$ is the intersection graph of the intervals $[f_j-1, \ell_j]$, $j = 1, 2, \ldots, n$.

Conversely, let $\mathcal{G}$ be the intersection graph of a set of n (closed) intervals I_j, $j = 1, 2, \ldots, n$. Let the distinct endpoints $e_1, e_2, \ldots, e_m$ of these intervals be ordered so that $e_1 < e_2 < \ldots < e_m$. Then the m by n incidence matrix $A = (a_{ij})$, defined by setting $a_{ij} = 1$ or 0 according as e_1 is or is not in I_j, has consecutive 1's in each column. Moreover, D is obtained from A by deleting rows. Hence D has the consecutive 1's property.

The incidence matrix of Fig. 7.3 has the consecutive 1's property; that of Fig. 7.4 does not. (A rearranged form of Fig. 7.3 is shown in Fig. 7.5.) Consequently the graph of Fig. 7.1 is an interval graph (its "intervals" being displayed in Fig. 7.5), but that of Fig. 7.2 is not.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Fig. 7.5

8. CONSECUTIVE 1's AND TOTAL UNIMODULARITY

A (0, 1)-matrix A is said to be totally unimodular [6] if each of its square submatrices has determinant ± 1 or 0. It follows from known sufficient conditions for the total unimodularity property that if A has the consecutive 1's property, then A is totally unimodular [6]. This fact can also be proved directly without difficulty by induction on the number of rows in a square matrix having the consecutive 1's property. The total unimodularity property has significant implications in linear inequality theory. In particular, if A is totally unimodular, and if b is a vector having integral components, then the convex polyhedron defined by the linear inequalities

$$(8.1) \qquad \begin{aligned} Au &\geq b \ , \\ u &\geq 0 \end{aligned}$$

has all integral extreme points. That is, if u is an extreme solution of (8.1), then u has integral components. Consequently, the linear program

$$(8.2) \qquad \begin{aligned} Au &\geq b \ , \\ u &\geq 0 \ , \end{aligned}$$

$$\text{minimize } c \cdot u$$

always has integral solution vectors u, provided A is totally unimodular and b is integral. In (8.2), c is a given real vector.

We can apply these facts and Theorem 7.1 to derive a certain combinatorial duality theorem for interval graphs which, though similar in some respects to known results about rigid circuit graphs [2], [7], does not hold for the latter. To state this theorem, we first require a defin-ition. We shall say that a set of vertices in a graph $\mathcal{G}$ represents all dominant cliques of $\mathcal{G}$ provided each dominant clique contains some vertex of the set. Our interest is in a minimum representation, that is, a set of vertices which represents all dominant cliques and has minimum cardinality over all such sets.

Theorem 8.1. Let $\mathscr{G}$ be an interval graph. Then
the minimum number of vertices of $\mathscr{G}$ required to represent
all dominant cliques is equal to the maximum number of
dominant cliques that are mutually disjoint.

Proof. Let D be the dominant clique vs. vertex incidence
matrix of $\mathscr{G}$. In view of Theorem 7.1 and the preceding
discussion, the problem of finding a minimum representation
of dominant cliques can be posed as that of solving the
linear program

$$(8.3) \qquad \begin{aligned} Du &\geq 1 , \\ u &\geq 0 , \end{aligned}$$

$$\text{minimize } 1 \cdot u .$$

Here 1 is a vector all of whose components are unity.
The dual of (8.3) is

$$(8.4) \qquad \begin{aligned} D^T w &\leq 1 , \\ w &\geq 0 , \end{aligned}$$

$$\text{maximize } 1 \cdot w .$$

Since D is totally unimodular, so is D^T, and thus the
program (8.4) also has $(0, 1)$–solution vectors w. Con–
sequently (8.4) asks for the maximum number of dominant
cliques that are pairwise disjoint. Thus Theorem 8.1
follows from the duality theorem for linear inequalities.

It can be seen similarly that if $\mathcal{G}$ is an interval
graph, then the maximum number of vertices, no two of which
are in the same dominant clique (i.e., the maximum number
of vertices, no two of which are joined by an edge) is
equal to the minimum number of dominant cliques that cover
all vertices (i.e., the minimum number of cliques that
cover all vertices). This theorem is in fact known to be
true more generally for rigid circuit graphs [2], [7]. But
Theorem 8.1 is false for rigid circuit graphs. The graph
of Fig. 7.2 is a case in point, since the minimum number
of vertices required to represent all dominant cliques is
two, whereas the maximum number of mutually disjoint
dominant cliques is one.

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